

A CONSISTENT MARKOV PARTITION PROCESS GENERATED FROM THE PAINTBOX PROCESS

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Abstract

We study a family of Markov processes on $\mathcal{P}^{(k)}$, the space of partitions of the natural numbers with at most k blocks. The process can be constructed from a Poisson point process on $\mathbb{R}^+ \times \prod_{i=1}^k \mathcal{P}^{(k)}$ with intensity $dt \otimes \varrho_\nu^{(k)}$, where ϱ_ν is the distribution of the paintbox based on the probability measure ν on \mathcal{P}_m , the set of ranked-mass partitions of 1, and $\varrho_\nu^{(k)}$ is the product measure on $\prod_{i=1}^k \mathcal{P}^{(k)}$. We show that these processes possess a unique stationary measure, and we discuss a particular set of reversible processes for which transition probabilities can be written down explicitly.

Keywords: paintbox process, Ewens partition, Poisson-Dirichlet distribution, partition process

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1. Introduction

Markov processes on the space of partitions appear in a variety of situations in scientific literature, such as, but not limited to, physical chemistry, astronomy, and population genetics. See Aldous [1] for a relatively recent overview of this literature. Well-behaved mathematically tractable models of random partitions are of interest to probabilists as well as statisticians and scientists, [10],[12],[16],[13]. Ewens [10] first introduced the Ewens sampling formula in the context of theoretical population biology. Kingman's [12] coalescent model was introduced as a model for population genetics, still its most natural setting. However, since the seminal work of Ewens and Kingman, random partitions have appeared in areas ranging from classification models, as in

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[7], [16], to probability theory, see [3],[17]. McCullagh [13] describes how the Ewens model can be used in the classical problem of estimating the number of unseen species, introduced by Fisher [11] and later studied by many, including Efron and Thisted [9].

Berestycki [2] studies a family of partition processes, called exchangeable fragmentation-coalescence (EFC) processes, whose paths are generated by a combination of independent coalescent and fragmentation processes. The mathematical tractability of coalescent and fragmentation processes has led to the development of many results for EFC processes and has led to interest in more complex models. For a sample of these results and relevant references see [3],[15],[17]. The study of processes, such as the EFC process, which admit a more general study of partition-valued processes is of interest from a theoretical as well as applied perspective. In this paper, we study a family of processes which is similar in spirit to the EFC process, but whose sample paths are quite different.

2. Preliminaries

Throughout this paper, \mathcal{P} denotes the space of set partitions of the natural numbers \mathbb{N} . We regard an element B of \mathcal{P} as a collection of disjoint non-empty subsets, called blocks, written $B = \{B_1, B_2, \dots\}$, such that $\bigcup_i B_i = \mathbb{N}$. The blocks are unordered, but, where necessary, they are listed in the order of their least element. We write $B = (B_1, B_2, \dots)$ whenever we wish to emphasize that blocks are listed in a particular order. For $B \in \mathcal{P}$ and $b \in B$, $\#B$ is the number of blocks of B and $\#b$ is the number of elements of b . For any $A \subset \mathbb{N}$, let $B|_A$ denote the restriction of B to A . Wherever necessary, $\mathcal{P}^{(k)}$ denotes the space of partitions of \mathbb{N} with at most k blocks, i.e. $\mathcal{P}^{(k)} := \{B \in \mathcal{P} : \#B \leq k\}$. For fixed $n \in \mathbb{N}$, $\mathcal{P}_{[n]}$ and $\mathcal{P}_{[n]}^{(k)}$ are the restriction to $[n] := \{1, \dots, n\}$ of \mathcal{P} and $\mathcal{P}^{(k)}$ respectively.

It is sometimes convenient to regard a partition B as either an equivalence relation defined by $B(i, j) = 1 \Leftrightarrow i \sim_B j$ or an $n \times n$ symmetric Boolean matrix whose (i, j) th entry is $B(i, j)$. These three representations are equivalent and we use the same notation to refer to any one of them.

For each $\pi, \pi' \in \mathcal{P}$, we define the metric $d : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ such that

$$d(\pi, \pi') = 1 / \max\{n \in \mathbb{N} : \pi|_{[n]} = \pi'|_{[n]}\}.$$

The space (\mathcal{P}, d) is compact [5].

In addition, we define the projection $D_{m,n} : \mathcal{P}_{[n]} \rightarrow \mathcal{P}_{[m]}$ for each $n \geq m \geq 1$ by $D_{m,n}B_{[n]} = B_{[n] \setminus [m]}$. In the matrix representation, $D_{m,n}B$ is the leading $m \times m$ submatrix of B . We seek processes $B := (B_t, t \geq 0)$ on \mathcal{P} such that for each $n \in \mathbb{N}$, the restriction of B to $[n]$, $B_{|[n]}$, is finitely exchangeable and consistent. That is,

- $\sigma(B_{|[n]}) \sim B_{|[n]}$ for each $\sigma \in \mathcal{S}_n$, the symmetric group acting on $[n]$, and
- $B_{[n] \setminus [m]} \sim B_{|[m]}$ for each $m < n$.

It is more convenient to work with \mathcal{P} as the state space of our process than the space $\mathcal{P}_m = \{(s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_i s_i \leq 1\}$ of ranked-mass partitions of $x \in [0, 1]$. In accordance with the notation for set partitions, let $\mathcal{P}_m^{(k)} := \{s \in \mathcal{P}_m : s_j = 0 \forall j > k, \sum_{i=1}^k s_i = 1\}$ denote the ranked k -simplex. There is an intimate relationship between exchangeable processes on \mathcal{P} and processes on \mathcal{P}_m through the paintbox process.

For $s \in \mathcal{P}_m$, let $X := (X_1, X_2, \dots)$ be independent random variables with distribution

$$\mathbb{P}_s(X_i = j) = \begin{cases} s_j, & j \geq 1 \\ 1 - \sum_{i=1}^\infty s_i, & j = -i \\ 0, & \text{o.w.} \end{cases}$$

The partition $\Pi(X)$ generated by s through X satisfies $i \sim_{\Pi(X)} j$ if and only if $X_i = X_j$. The distribution of $\Pi(X)$ is written ϱ_s and $\Pi(X)$ is called the paintbox based on s . For a measure ν on \mathcal{P}_m , the paintbox based on ν is the ν -mixture of paintboxes, written $\varrho_\nu(\cdot) := \int_{\mathcal{P}_m} \varrho_s(\cdot) \nu(ds)$. Any partition obtained in this way is an exchangeable random partition of \mathbb{N} and every infinitely exchangeable partition admits a representation as the paintbox generated by some ν . See [5] and [17] for more details on the paintbox process.

We are particularly interested in exchangeable Markovian transition probabilities (p_n) , where, for every n , p_n is a transition probability on $\mathcal{P}_{[n]}$ which satisfies

$$p_n(B, B') = \sum_{B'' \in D_{n,n+1}^{-1}(B')} p_{n+1}(B^*, B''), \quad (2.1)$$

for each $B, B' \in \mathcal{P}_{[n]}$ and $B^* \in D_{n,n+1}^{-1}(B)$. Burke and Rosenblatt [8] show that (2.1) is necessary and sufficient for (p_n) to be consistent under selection from \mathbb{N} .

Likewise, for a continuous-time Markov process, $(B_n(t), t \geq 0)_{n \in \mathbb{N}}$, where $B_n(t)$ is a process on $\mathcal{P}_{[n]}$ with infinitesimal generator Q_n , it is sufficient that the entries of Q_n satisfy (2.1) for there to be a Markov process on \mathcal{P} with those finite-dimensional transition rates.

3. The ϱ_ν -Markov chain on $\mathcal{P}^{(k)}$

Let $n, k \in \mathbb{N}$ and let ν be a probability measure on the ranked k -simplex $\mathcal{P}_m^{(k)}$, so that the paintbox based on ν is obtained by a conditionally i.i.d. sample from ν , i.e. given $s \sim \nu$, X_1, X_2, \dots are i.i.d. with $\mathbb{P}_s(X_i = j) = s_j$ for each $j = 1, \dots, k$. For convenience, we write $B \in \mathcal{P}^{(k)}$ as an ordered list (B_1, \dots, B_k) where B_i corresponds to the i th block of B in order of appearance for $i \leq \#B$ and $B_i = \emptyset$ for $i = \#B + 1, \dots, k$.

Consider the following Markov transition operation $B \mapsto B'$ on $\mathcal{P}^{(k)}$. Let $B = (B_1, \dots, B_k) \in \mathcal{P}^{(k)}$ and, independently of B , generate C_1, C_2, \dots which are independent and identically distributed accord to ϱ_ν . For each i , we write $C_i := (C_{i1}, \dots, C_{ik}) \in \mathcal{P}^{(k)}$. Independently of B, C_1, C_2, \dots , generate $\sigma_1, \sigma_2, \dots$ which are independent uniform random permutations of $[k]$. Given $\sigma := (\sigma_1, \sigma_2, \dots, \sigma_k)$, we arrange B, C_1, \dots, C_k in matrix form as follows:

$$\begin{array}{c} \begin{array}{cccc} C_{\cdot 1} & C_{\cdot 2} & \dots & C_{\cdot k} \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_k \end{array} \begin{pmatrix} C_{1, \sigma_1(1)} \cap B_1 & C_{1, \sigma_1(2)} \cap B_1 & \dots & C_{1, \sigma_1(k)} \cap B_1 \\ C_{2, \sigma_2(1)} \cap B_2 & C_{2, \sigma_2(2)} \cap B_2 & \dots & C_{2, \sigma_2(k)} \cap B_2 \\ \vdots & \vdots & \ddots & \vdots \\ C_{k, \sigma_k(1)} \cap B_k & C_{k, \sigma_k(2)} \cap B_k & \dots & C_{k, \sigma_k(k)} \cap B_k \end{pmatrix} =: B \cap C^\sigma. \end{array}$$

$B \cap C^\sigma$ is a matrix with row totals corresponding to the blocks of B and column totals $C_{\cdot j} = \bigcup_{i=1}^k (C_{i, \sigma_i(j)} \cap B_i)$. Finally, B' is obtained as the collection of non-empty blocks of $(C_{\cdot 1}, \dots, C_{\cdot k})$. The non-empty entries of $B \cap C^\sigma$ form a partition in $\mathcal{P}^{(k^2)}$ which corresponds to the greatest lower bound $B \wedge B'$.

Proposition 3.1. *The above description gives rise to finite-dimensional transition probabilities on $\mathcal{P}_{[n]}^{(k)}$*

$$p_n(B, B'; \nu) = \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!} \varrho_\nu(B'_b). \quad (3.1)$$

Proof. Let $A \in \mathcal{P}^{(k)}$. Fix $n, k \in \mathbb{N}$, put $B := A_{|[n]} \in \mathcal{P}_{[n]}^{(k)}$. Let C_1, \dots, C_k be i.i.d. ϱ_ν -distributed partitions and $\sigma := (\sigma_1, \dots, \sigma_k)$ i.i.d. uniform random permutations of $[k]$ as described above. Let B' be the set partition obtained from the column totals of the matrix $B \cap C^\sigma$ in the above construction.

From the matrix construction, it is clear that for each $i = 1, \dots, k$, the restriction $B'_{|B_i}$ is equal to the set partition in $\mathcal{P}_{[n]}^{(k)}$ associated with $C_i[B_i] := (C_{i1} \cap B_i, \dots, C_{ik} \cap B_i)$. Conversely, the transition $B \mapsto B'$ occurs only if the collection (C_1, \dots, C_k) is such that, for each $B_i \in B$, $C_i[B_i] = B'_{|B_i}$. By consistency of the paintbox process, for each $i = 1, \dots, k$, $C_i[B_i]$ has probability

$$\varrho_\nu(C_i[B_i]) = \varrho_\nu(B'_{|B_i}).$$

Independence of the C_i implies that the probability of $B \wedge B'$ given B is

$$\prod_{b \in B} \varrho_\nu(B'_b).$$

Finally, each uniform permutation σ_i has probability $1/k!$ and there are $\frac{k!}{(k - \#B')!} \prod_{b \in B} (k - \#B'_b)!$ collections $\sigma_1, \dots, \sigma_{\#B}$ such that the column totals of $B \cap C^\sigma$ correspond to the blocks of B' . This completes the proof.

For fixed n , (3.1) only depends on B and B' through ϱ_ν and the number of blocks of B and B' and is, therefore, finitely exchangeable. I appeal to (2.1) to establish consistency.

Proposition 3.2. *For any measure ν on $\mathcal{P}_m^{(k)}$, let $(p_n(\cdot, \cdot; \nu))_{n \geq 1}$ be the collection of transition probabilities on $\mathcal{P}_{[n]}^{(k)}$ defined in (3.1). Then (p_n) is a consistent family of transition probabilities.*

Proof. Fix $n, k \in \mathbb{N}$ and let $B, B' \in \mathcal{P}_{[n]}^{(k)}$. To establish consistency it is enough to verify condition (2.1) from theorem 1 of [8], i.e. for each ν and $B^* \in D_{n,n+1}^{-1}(B)$,

$$p_{n+1}(B^*, D_{n,n+1}^{-1}(B'); \nu) = p_n(B, B'; \nu).$$

We assume without loss of generality that $B^* \in D^{-1}(B)$ is obtained from B by the operation $n+1 \mapsto B_1 \in B$ and we write $B_1^* := B_1 \cup \{n+1\}$. Likewise, for $B'' \in D_{n,n+1}^{-1}(B')$ obtained by $n+1 \mapsto B'_i \in B' \cup \{\emptyset\}$, write $B_i'^* := B'_i \cup \{n+1\}$. So either $n+1 \in B_i'^*$ for some $i = 1, \dots, \#B'$ or $n+1$ is inserted in B' as a singleton.

The change to $B \cap C^\sigma$ that results from inserting $n+1$ into $B_1 \in B$ and $B'_i \in B'$ is summarized by the following matrix. Note that $B'_j = \emptyset$ for $j > \#B'$.

$$\begin{array}{c} B'_1 \quad B'_2 \quad \dots \quad B'_i^* \quad \dots \quad B'_k \\ B_1^* \left(\begin{array}{cccccc} B'_1 \cap B_1 & B'_2 \cap B_1 & \dots & (B'_i \cap B_1) \cup \{n+1\} & \dots & B'_k \cap B_1 \\ B'_1 \cap B_2 & B'_2 \cap B_2 & \dots & B'_i \cap B_2 & \dots & B'_k \cap B_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B'_1 \cap B_k & B'_2 \cap B_k & \dots & B'_i \cap B_k & \dots & B'_k \cap B_k \end{array} \right) \end{array}.$$

Here, the blocks of B are listed in any order, with empty sets inserted as needed, and the blocks of B' are listed in order of least elements, with $k - \#B'$ empty sets at the end.

Given B' , the set of compatible partitions $D_{n,n+1}^{-1}(B')$ consists of three types depending on the subset $B_1 \subset [n]$ and the block of B' into which $\{n+1\}$ is inserted. Let $B'' \in D_{n,n+1}^{-1}(B')$ be the partition of $[n+1]$ obtained by inserting $n+1$ in B' . Either

- (i) $n+1$ is inserted into a block B'_i such that $B'_i \cap B_1 \neq \emptyset \Rightarrow \#B''_{|B_1^*} = \#B'_{|B_1}$,
- (ii) $n+1$ is inserted into a block $B'_i \neq \emptyset$ such that $B'_i \cap B_1 = \emptyset \Rightarrow \#B''_{|B_1^*} = \#B'_{|B_1} + 1$,
or
- (iii) $n+1$ is inserted into B' as a singleton block $\Rightarrow \#B''_{|B_1^*} = \#B'_{|B_1} + 1$ and $\#B'' = \#B' + 1$; we denote this partition by B'_\emptyset .

There are $k - \#B'$ empty columns in which $\{n+1\}$ can be inserted as a singleton in B' , as in (iii). For B'' obtained by (ii), the restriction of B'' to B_1^* coincides with the restriction of B'_\emptyset to B_1^* , so each of these restrictions has the same probability under ϱ_ν . For notational convenience in the following calculation, let \mathcal{D}_1 be those elements of $D_{n,n+1}^{-1}(B')$ which satisfy condition (i) above and \mathcal{D}_2 those which satisfy condition (ii).

$$p_{n+1}(B^*, D_{n,n+1}^{-1}(B'); \nu) = \sum_{B'' \in D_{n,n+1}^{-1}(B')} \frac{k!}{(k - \#B'')!} \prod_{b \in B^*} \frac{(k - \#B''_b)!}{k!} \varrho_\nu(B''_b) \quad (3.2)$$

$$= \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!} \left[\sum_{B'' \in \mathcal{D}_1} \prod_{b \in B^*} \varrho_\nu(B''_b) + \sum_{B'' \in \mathcal{D}_2} \frac{1}{k - \#B'_{|B_1}} \prod_{b \in B^*} \varrho_\nu(B''_b) + \frac{k - \#B'}{k - \#B'_{|B_1}} \prod_{b \in B^*} \varrho_\nu(B'_{\emptyset|b}) \right] \quad (3.3)$$

$$= \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!} \prod_{b \in B^*: b \neq B_1^*} \varrho_\nu(B'_b) \left[\sum_{B'' \in \mathcal{D}_1} \varrho_\nu(B''_{|B_1^*}) + \sum_{B'' \in \mathcal{D}_2} \frac{1}{k - \#B'_{|B_1}} \varrho_\nu(B''_{|B_1^*}) + \frac{k - \#B'}{k - \#B'_{|B_1}} \varrho_\nu(B'_{\emptyset|B_1^*}) \right] \quad (3.4)$$

$$= \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!} \prod_{b \in B: b \neq B_1^*} \varrho_\nu(B'_b) \left[\sum_{B'' \in D_{\#B_1, \#B_1+1}^{-1}(B'_{|B_1})} \varrho_\nu(B'') \right] \quad (3.5)$$

$$= \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!} \prod_{b \in B: b \neq B_1^*} \varrho_\nu(B'_b) [\varrho_\nu(B'_{|B_1})] \quad (3.6)$$

$$= \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!} \prod_{b \in B} \varrho_\nu(B'_b) \\ = p_n(B, B'; \nu).$$

Here, (3.3) is obtained from (3.2) by factoring $\frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_b)!}{k!}$ out of the sum and using observations (i), (ii) and (iii). In (3.4), we use the fact that for any $B'' \in \mathcal{D}_2$, $B''_{|B_1^*} = B'_{\emptyset|B_1^*}$, and there are $\#B' - \#B'_{|B_1}$ elements in \mathcal{D}_2 according to (ii). Line (3.5) follows by observing that each $B'' \in \mathcal{D}_1$ corresponds to an element of $D_{\#B_1, \#B_1+1}^{-1}(B'_{|B_1})$ and $B'_{\emptyset|B_1^*}$ is the element of $D_{\#B_1, \#B_1+1}^{-1}(B'_{|B_1})$ obtained by inserting $\{n+1\}$ as a singleton in $B'_{|B_1}$. Finally, (3.6) follows from (3.5) by consistency of the paintbox process. This completes the proof.

The following result is immediate by finite exchangeability and consistency of (3.1) for every n and Kolmogorov's extension theorem (theorem 36.1, [6]).

Theorem 3.1. *There exists a transition probability $p(\cdot, \cdot; \nu)$ on $(\mathcal{P}^{(k)}, \sigma(\bigcup_n \mathcal{P}_{[n]}^{(k)}))$ whose finite-dimensional restrictions are given by (3.1).*

We call the discrete-time process governed by $p(\cdot, \cdot; \nu)$ the ϱ_ν -Markov chain with state space $\mathcal{P}^{(k)}$.

3.1. Equilibrium measure

From (3.1), it is clear that for each $n, k \in \mathbb{N}$ and $B, B' \in \mathcal{P}_{[n]}^{(k)}$, $p_n(B, B'; \nu)$ is strictly positive provided ν is such that $\nu(s) > 0$ for some $s = (s_1, \dots, s_k) \in \mathcal{P}_m^{(k)}$ with $s_k > 0$. Under this condition, the finite-dimensional chains are aperiodic and irreducible on $\mathcal{P}_{[n]}^{(k)}$ and, therefore, have a unique stationary distribution. In fact, the finite-dimensional chains based on ν are aperiodic and irreducible provided ν is not degenerate at $(1, 0, \dots, 0) \in \mathcal{P}_m^{(k)}$. The existence of a unique stationary distribution for each n implies that there is a unique stationary probability measure on $(\mathcal{P}^{(k)}, \sigma(\bigcup_n \mathcal{P}_{[n]}^{(k)}))$ for $p(\cdot, \cdot; \nu)$ from theorem 3.1.

Proposition 3.3. *Let ν be a measure on $\mathcal{P}_m^{(k)}$ such that ν is non-degenerate at $(1, 0, \dots, 0) \in \mathcal{P}_m^{(k)}$. Then there exists a unique stationary distribution $\theta_n(\cdot; \nu)$ for $p_n(\cdot, \cdot; \nu)$ for each $n \geq 1$.*

Proof. Fix $n \in \mathbb{N}$ and let ν be any measure on $\mathcal{P}_m^{(k)}$ other than that which puts unit mass at $(1, 0, \dots, 0)$. For $B = (B_1, \dots, B_m) \in \mathcal{P}_{[n]}^{(k)}$, (3.1) gives the transition probability

$$p_n(B, B; \nu) = \frac{k!}{(k-m)!} \prod_{i=1}^m \frac{1}{k} \varrho_\nu(B_i)$$

and $\varrho_\nu(B_i) = \varrho_\nu([\#B_i]) > 0$ for each $i = 1, \dots, m$. Hence, $p_n(B, B; \nu) > 0$ for every $B \in \mathcal{P}_{[n]}^{(k)}$ and the chain is aperiodic.

To see that the chain is irreducible, let $B, B' \in \mathcal{P}_{[n]}^{(k)}$ and let 1_n denote the one block partition of $[n]$. Then

$$p_n(B, 1_n; \nu) = k \prod_{b \in B} \frac{1}{k} \varrho_\nu([\#b]) > 0$$

and, since ν is not degenerate at $(1, 0, \dots, 0)$, there exists a path $1_n \mapsto B'$ by recursively partitioning 1_n until it coincides with B' . For instance, let $B' := (B'_1, \dots, B'_m) \in \mathcal{P}^{(k)}$.

One such path from 1_n to B' is

$$1_n \rightarrow (B'_1, \bigcup_{i=2}^m B'_i) \rightarrow (B'_1, B'_2, \bigcup_{i=3}^m B'_i) \rightarrow \cdots \rightarrow B'$$

which has positive probability for any non-degenerate ν . Hence $p_n(\cdot, \cdot; \nu)$ is irreducible, which establishes the existence of a unique stationary distribution for each n .

Theorem 3.2. *Let ν be a measure on $\mathcal{P}_m^{(k)}$ such that $\nu((1, 0, \dots, 0)) < 1$. Then there exists a unique stationary probability measure $\theta(\cdot; \nu)$ for the ϱ_ν -Markov chain on $\mathcal{P}^{(k)}$.*

Proof. For ν satisfying this condition, proposition 3.3 shows that a stationary distribution exists for each $n \geq 1$. Let $(\theta_n(\cdot; \nu), n \geq 1)$ be the collection of stationary distributions for the finite-dimensional transition probabilities $(p_n(\cdot, \cdot; \nu), n \geq 1)$. We now show that the θ_n are consistent and finitely exchangeable for each n .

Fix $n \in \mathbb{N}$ and let $B \in \mathcal{P}_{[n]}^{(k)}$. Then stationarity of $\theta_n(\cdot; \nu)$ implies

$$\sum_{B' \in \mathcal{P}_{[n]}^{(k)}} \theta_n(B'; \nu) p_n(B', B; \nu) = \theta_n(B; \nu).$$

Now write $\theta_n(\cdot) \equiv \theta_n(\cdot; \nu)$ and $p_n(\cdot, \cdot) \equiv p_n(\cdot, \cdot; \nu)$ for convenience and let $B' \in \mathcal{P}_{[n]}^{(k)}$.

We have

$$\begin{aligned} \underbrace{\sum_{B'' \in D_{n,n+1}^{-1}(B')} \theta_{n+1}(B'')}_{(\theta_{n+1} D_{n,n+1}^{-1})(B')} &= \sum_{B'' \in D_{n,n+1}^{-1}(B')} \sum_{B^* \in \mathcal{P}_{[n+1]}^{(k)}} \theta_{n+1}(B^*) p_{n+1}(B^*, B'') \\ &= \sum_{B^* \in \mathcal{P}_{[n+1]}^{(k)}} \theta_{n+1}(B^*) \left[\sum_{B'' \in D_{n,n+1}^{-1}(B')} p_{n+1}(B^*, B'') \right] \\ &= \sum_{B \in \mathcal{P}_{[n]}^{(k)}} \sum_{B^* \in D_{n,n+1}^{-1}(B)} \theta_{n+1}(B^*) [p_n(B, B')] \\ &= \sum_{B \in \mathcal{P}_{[n]}^{(k)}} p_n(B, B') \sum_{B^* \in D_{n,n+1}^{-1}(B)} \theta_{n+1}(B^*) \\ &= \sum_{B \in \mathcal{P}_{[n]}^{(k)}} p_n(B, B') (\theta_{n+1} D_{n,n+1}^{-1})(B). \end{aligned}$$

So we have that $\theta_{n+1} D_{n,n+1}^{-1}$ is stationary for p_n which implies that $\theta_n \equiv \theta_{n+1} D_{n,n+1}^{-1}$ by uniqueness and θ_n is consistent for each n .

Let σ be a permutation of $[n]$. Then for any $B, B' \in \mathcal{P}_{[n]}^{(k)}$, $p_n(\sigma(B), \sigma(B')) = p_n(B, B')$ by exchangeability of p_n . It follows that θ_n is finitely exchangeable for each n since

$$\sum_{B \in \mathcal{P}_{[n]}^{(k)}} \theta_n(\sigma(B)) p_n(\sigma(B), \sigma(B')) = \theta_n(\sigma(B'))$$

by stationarity, and $p_n(\sigma(B), \sigma(B')) = p_n(B, B')$ implies that

$$\sum_{B \in \mathcal{P}_{[n]}^{(k)}} \theta_n(\sigma(B)) p_n(B, B') = \theta_n(\sigma(B')).$$

Hence, $\theta_n \circ \sigma$ is stationary for p_n and $\theta_n \equiv \theta_n \circ \sigma$ by uniqueness.

Kolmogorov consistency implies that there exists a unique exchangeable stationary probability measure θ on $\mathcal{P}^{(k)}$ whose restriction to $[n]$ is θ_n for each $n \in \mathbb{N}$. This completes the proof.

4. The ϱ_ν -Markov process in continuous time

Let $\lambda > 0$, ν be a measure on $\mathcal{P}_m^{(k)}$ and for each $n \in \mathbb{N}$ define Markovian infinitesimal jump rates for a Markov process on $\mathcal{P}_{[n]}^{(k)}$ by

$$q_n(B, B'; \nu) = \begin{cases} \lambda p_n(B, B'; \nu), & B \neq B' \\ 0, & \text{o.w.} \end{cases} \quad (4.1)$$

where p_n is as in (3.1). The infinitesimal generator, Q_n^ν , of the process on $\mathcal{P}_{[n]}^{(k)}$ governed by q_n has entries

$$Q_n^\nu(B, B') = \lambda \times \begin{cases} p_n(B, B'; \nu), & B \neq B' \\ p_n(B, B; \nu) - 1, & B = B'. \end{cases} \quad (4.2)$$

We now construct a Markov process $B := (B(t), t \geq 0)$ in continuous time whose finite-dimensional transition rates are given by (4.1).

Definition 4.1. A process $B := (B(t), t \geq 0)$ on $\mathcal{P}^{(k)}$ is a ϱ_ν -Markov process if, for each $n \in \mathbb{N}$, $B|_{[n]}$ is a Markov process on $\mathcal{P}_{[n]}^{(k)}$ with Q -matrix Q_n^ν as in (4.2).

A process on $\mathcal{P}^{(k)}$ whose finite-dimensional restrictions are governed by Q_n^ν can be constructed according to the matrix construction from section 3 by permitting only transitions $B \mapsto B'$ for $B' \neq B$, where $B, B' \in \mathcal{P}_{[n]}^{(k)}$, and adding a hold time which is exponentially distributed with mean $-1/Q_n^\nu(B, B)$.

Proposition 4.1. *For a measure ν on $\mathcal{P}_m^{(k)}$, let $(Q_n^\nu)_{n \in \mathbb{N}}$ be the collection of Q -matrices in (4.2). For every $n \in \mathbb{N}$, the entries of Q_n^ν satisfy (2.1).*

Proof. Fix $n \in \mathbb{N}$ and let $B, B' \in \mathcal{P}_{[n]}^{(k)}$ such that $B \neq B'$. Then

$$Q_n^\nu(B, B') = \sum_{B'' \in D_{n,n+1}^{-1}(B')} Q_{n+1}^\nu(B_*, B'')$$

for all $B_* \in D_{n,n+1}^{-1}(B)$ by the consistency of p_n from proposition 3.2.

For $B' = B$ and $B_* \in D_{n,n+1}^{-1}(B)$, we have

$$\begin{aligned} \sum_{B'' \in D_{n,n+1}^{-1}(B)} Q_{n+1}^\nu(B_*, B'') &= \\ Q_{n+1}^\nu(B_*, B_*) + \sum_{B'' \in D_{n,n+1}^{-1}(B) \setminus \{B_*\}} Q_{n+1}^\nu(B_*, B'') &= \\ \lambda \left[p_{n+1}(B_*, B_*; \nu) - 1 + \sum_{B'' \in D_{n,n+1}^{-1}(B) \setminus \{B_*\}} p_{n+1}(B_*, B''; \nu) \right] &= \\ \lambda \left[\sum_{B'' \in D_{n,n+1}^{-1}(B)} p_{n+1}(B_*, B''; \nu) - 1 \right] &= \\ \lambda(p_n(B, B; \nu) - 1) &= \\ Q_n^\nu(B, B). \end{aligned}$$

Theorem 4.1. *For each measure ν on $\mathcal{P}_m^{(k)}$, there exists a Markov process $(B(t), t \geq 0)$ on $\mathcal{P}^{(k)}$ which has finite-dimensional transition rates given in (4.1).*

Proof. Let ν be a measure on $\mathcal{P}_m^{(k)}$ and $(B_{[n]}(t), t \geq 0)_{n \in \mathbb{N}}$ be the collection of restrictions of a ϱ_ν -Markov process with consistent Q -matrices $(Q_n^\nu)_{n \in \mathbb{N}}$ as in (4.2). For each n , Q_n^ν is finitely exchangeable and consistent with Q_{n+1}^ν by proposition 4.1, which is sufficient for $B_{[n]}$ to be consistent with $B_{[n+1]}$ for every n . Kolmogorov's extension theorem implies that there exist transition rates, Q^ν , on $\mathcal{P}^{(k)}$ such that for every $B, B' \in \mathcal{P}_{[n]}^{(k)}$,

$$Q_n^\nu(B, B') = Q^\nu(B_*, \{B'' \in \mathcal{P}^{(k)} : B''_{[n]} = B'\}),$$

for every $B_* \in \{B'' \in \mathcal{P}^{(k)} : B''_{[n]} = B\}$.

Finally, for every $B \in \mathcal{P}_{[n]}^{(k)}$, $Q_n^\nu(B, \mathcal{P}_{[n]}^{(k)} \setminus \{B\}) = \lambda(1 - p_n(B, B; \nu)) < \infty$ so that the sample paths of $B_{[n]}$ are càdlàg for every n , which implies that B is càdlàg.

Corollary 4.1. *For ν which satisfies the condition of theorem 3.2, the continuous-time process $B := (B(t), t \geq 0)$ with finite-dimensional rates $q_n(\cdot, \cdot; \nu)$ in (4.1) has unique stationary distribution $\theta(\cdot; \nu)$ from theorem 3.2.*

Proof. For each $n \in \mathbb{N}$, let $\theta_n(\cdot; \nu)$ be the unique finite-dimensional stationary distribution of $p_n(\cdot, \cdot; \nu)$ from (3.1). It is easy to verify that for each $n \in \mathbb{N}$, $\Theta_n^\nu := (\theta_n(B; \nu), B \in \mathcal{P}_{[n]}^{(k)})$ satisfies

$$(\Theta_n^\nu)^t Q_n^\nu = 0,$$

which establishes that Θ_n^ν is stationary for Q_n^ν for every n . The rest follows by theorem 3.2.

4.1. Poissonian construction

From the matrix construction at the beginning of section 3, a consistent family of finite-dimensional Markov processes with transition rates as in (4.1) can be constructed by a Poisson point process on $\mathbb{R}^+ \times \prod_{i=1}^k \mathcal{P}^{(k)}$ as follows. Let $P = \{(t, C_1, \dots, C_k)\} \subset \mathbb{R}^+ \times \prod_{i=1}^k \mathcal{P}^{(k)}$ be a Poisson point process with intensity measure $dt \otimes \lambda \varrho_\nu^{(k)}$ for some measure ν on $\mathcal{P}_m^{(k)}$ and $\lambda > 0$, where $\varrho_\nu^{(k)}$ is the product measure $\varrho_\nu \otimes \dots \otimes \varrho_\nu$ on $\prod_{i=1}^k \mathcal{P}^{(k)}$.

Construct an exchangeable process $B := (B(t), t \geq 0)$ on $\mathcal{P}^{(k)}$ by taking $\pi \in \mathcal{P}^{(k)}$ to be some exchangeable random partition and setting $B(0) = \pi$.

For each $n \in \mathbb{N}$, put $B_{|[n]}(0) = \pi_{|[n]}$ and

- if t is not an atom time for P , then $B_{|[n]}(t) = B_{|[n]}(t-)$;
- if t is an atom time for P so that $(t, C_1, \dots, C_k) \in P$, then, independently of $(B(s), s < t)$ and (t, C_1, \dots, C_k) generate $\sigma_1, \dots, \sigma_k$ i.i.d. uniform random permutations of $[k]$ and construct B' from the set partition induced by the column totals $(C_{\cdot,1}, \dots, C_{\cdot,k})$ of

$$\begin{array}{cccc} & C_{\cdot,1} & C_{\cdot,2} & \dots & C_{\cdot,k} \\ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_k \end{array} & \left(\begin{array}{cccc} C_{1,\sigma_1(1)} \cap B_1 & C_{1,\sigma_1(2)} \cap B_1 & \dots & C_{1,\sigma_1(k)} \cap B_1 \\ C_{2,\sigma_2(1)} \cap B_2 & C_{2,\sigma_2(2)} \cap B_2 & \dots & C_{2,\sigma_2(k)} \cap B_2 \\ \vdots & \vdots & \ddots & \vdots \\ C_{k,\sigma_k(1)} \cap B_k & C_{k,\sigma_k(2)} \cap B_k & \dots & C_{k,\sigma_k(k)} \cap B_k \end{array} \right) & =: & B \cap C^\sigma. \end{array}$$

where (B_1, \dots, B_k) are the blocks of $B = B_{[n]}(t-)$ listed in order of their least element, with $k - \#B$ empty sets at the end of the list.

- if $B' \neq B$, then $B_{[n]}(t) = B'$;
- if $B' = B$, $B_{[n]}(t) = B_{[n]}(t-)$.

Proposition 4.2. *The above process B is a Markov process on $\mathcal{P}^{(k)}$ with transition matrix Q^ν defined by theorem 4.1.*

Proof. This is clear from the consistency of both the paintbox process ϱ_ν and the Q_n^ν -matrices for every n and the fact that, by this construction, for any n such that $B_{[n]}(t) = \pi$ then $B_{[n][m]}(t) = D_{m,n}(\pi)$ for all $m < n$ and $B_{[p]}(t) \in D_{n,p}^{-1}(\pi)$ for all $p > n$.

Let \mathbb{P}_t be the semi-group of a ϱ_ν -Markov process $B(\cdot)$, i.e. for any continuous $\varphi : \mathcal{P}^{(k)} \rightarrow \mathbb{R}$

$$\mathbb{P}_t \varphi(\pi) := \mathbb{E}_\pi \varphi(B(t)),$$

the expectation of $\varphi(B(t))$ given $B(0) = \pi$.

Corollary 4.2. *A ϱ_ν -Markov process has the Feller property, i.e.*

- for each continuous function $\varphi : \mathcal{P}^{(k)} \rightarrow \mathbb{R}$, for each $\pi \in \mathcal{P}$ one has

$$\lim_{t \downarrow 0} \mathbb{P}_t \varphi(\pi) = \varphi(\pi),$$

- for all $t > 0$, $\pi \mapsto \mathbb{P}_t \varphi(\pi)$ is continuous.

Proof. The proof follows the same program as the proof of corollary 6 in [2].

Let $C_f := \{f : \mathcal{P}^{(k)} \rightarrow \mathbb{R} : \exists n \in \mathbb{N} \text{ s.t. } \pi_{[n]} = \pi'_{[n]} \Rightarrow f(\pi) = f(\pi')\}$ be a set of functions which is dense in the space of continuous functions from $\mathcal{P}^{(k)} \rightarrow \mathbb{R}$. It is clear that for $g \in C_f$, $\lim_{t \downarrow 0} \mathbb{P}_t g(\pi) = g(\pi)$ since the first jump-time of $B(\cdot)$ is an exponential variable with finite mean. The first point follows for all continuous functions $\mathcal{P}^{(k)} \rightarrow \mathbb{R}$ by denseness of C_f .

For the second point, let $\pi, \pi' \in \mathcal{P}^{(k)}$ such that $d(\pi, \pi') < 1/n$ and use the same Poisson point process P to construct two ϱ_ν -Markov processes, $B(\cdot)$ and $B'(\cdot)$, with starting points π and π' respectively. By the construction, $B_{[n]} = B'_{[n]}$ and $d(B(t), B'(t)) < 1/n$ for all $t \geq 0$. It follows that for any continuous g , $\pi \mapsto \mathbb{P}_t g(\pi)$ is continuous.

This allows us to characterize the ϱ_ν -Markov process in terms of its infinitesimal generator. Let $B := (B(t), t \geq 0)$ be the ϱ_ν -Markov process on $\mathcal{P}^{(k)}$ with transition rates characterized by $(q_n)_{n \in \mathbb{N}}$ as in (4.1). The infinitesimal generator, \mathcal{A} , of B is given by

$$\mathcal{A}(f)(\pi) = \int_{\mathcal{P}^{(k)}} f(\pi') - f(\pi) Q^\nu(\pi, d\pi'),$$

for every $f \in C_f$.

5. Asymptotic frequencies

Definition 5.1. A subset $A \subset \mathbb{N}$ is said to have asymptotic frequency λ if

$$\lambda := \lim_{n \rightarrow \infty} \frac{\#\{i \leq n : i \in A\}}{n} \quad (5.1)$$

exists, and a random partition $B := (B_1, B_2, \dots) \in \mathcal{P}$ is said to have asymptotic frequencies if each block of B has asymptotic frequency almost surely.

Adopting the notation of Berestycki [2], let $\Lambda(B) = (\|B_1\|, \|B_2\|, \dots)^\downarrow$ be the decreasing arrangement of asymptotic frequencies of a partition $B = (B_1, B_2, \dots) \in \mathcal{P}$ which possesses asymptotic frequencies, some of which could be 0.

According to Kingman's representation theorem (theorem 2.2, [17]) any exchangeable random partition of \mathbb{N} possesses asymptotic frequencies. Intuitively, this is a consequence of generating an exchangeable random partition of \mathbb{N} by the paintbox process.

The process described in section 3 only assigns positive probability to transitions involving two partitions with at most k blocks. From the Poissonian construction of the transition rates in section 4.1 it is evident that the states of $B = (B(t), t \geq 0)$ will have at most k blocks almost surely. Moreover, the description of the transition rates in terms of the paintbox process allows us to describe the associated measure-valued process of $B := (B(t), t \geq 0)$ characterized by λ and ν .

5.1. Poissonian construction

Consider the following Poissonian construction of a measure-valued process $X := (X(t), t \geq 0)$ on $\mathcal{P}_m^{(k)}$. For any $k \in \mathbb{N}$, $\lambda > 0$ and ν as above, let $P' = \{(t, P'_1, \dots, P'_k)\} \subset$

$\mathbb{R}^+ \times \prod_{i=1}^k \mathcal{P}_m^{(k)}$ be a Poisson point process with intensity measure $dt \otimes \lambda\nu^{(k)}$, where $\nu^{(k)}$ is the product measure $\nu \otimes \dots \otimes \nu$ on $\prod_{i=1}^k \mathcal{P}_m^{(k)}$.

Construct a process $X := (X(t), t \geq 0)$ on $\mathcal{P}_m^{(k)}$ by generating p_0 from some probability distribution on $\mathcal{P}_m^{(k)}$. Put $X(0) = p_0$ and

- if t is not an atom time for P' , then $X(t) = X(t-)$;
- if t is an atom time for P' so that $(t, P'_1, \dots, P'_k) \in P'$, with $P'_j = (P_1^j, \dots, P_k^j)$ for each $j = 1, \dots, k$, and $X(t-) = (x_1, \dots, x_k) \in \mathcal{P}_m^{(k)}$, then, independently of $(X(s), s < t)$ and (t, P'_1, \dots, P'_k) , generate $\sigma_1, \dots, \sigma_k$ i.i.d. uniform random permutations of $[k]$ and construct $X(t)$ from the marginal column totals of

$$\begin{array}{cccc} P_1^\bullet & P_2^\bullet & \dots & P_k^\bullet \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{matrix} \begin{pmatrix} x_1 P_{\sigma_1(1)}^1 & x_1 P_{\sigma_1(2)}^1 & \dots & x_1 P_{\sigma_1(k)}^1 \\ x_2 P_{\sigma_2(1)}^2 & x_2 P_{\sigma_2(2)}^2 & \dots & x_2 P_{\sigma_2(k)}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_k P_{\sigma_k(1)}^k & x_k P_{\sigma_k(2)}^k & \dots & x_k P_{\sigma_k(k)}^k \end{pmatrix} \end{array}.$$

i.e. put $X(t) = (P_1^\bullet, P_2^\bullet, \dots, P_k^\bullet)^\downarrow := \left(\sum_{i=1}^k x_i P_{\sigma_i(j)}^i, 1 \leq j \leq k \right)^\downarrow$.

Theorem 5.1. *Let $X := (X(t), t \geq 0)$ be the process constructed above. Then $X =_\mathcal{L} \Lambda(B)$ where $B := (B(t), t \geq 0)$ is the ϱ_ν -Markov process from theorem 4.1.*

Proof. Fix $k \in \mathbb{N}$ and let $\nu(\cdot)$ be a measure on $\mathcal{P}_m^{(k)}$.

In the description of the sample paths of B in section 4, note that generating $(C_1, \dots, C_k) \sim \varrho_\nu^{(k)}$ is equivalent to first generating $s_i \sim \nu$ independently for each $i = 1, \dots, k$, then generating random partitions C_i by sampling from s_i for each $i = 1, \dots, k$. Finally, B'_i is set equal to the marginal total of column i of the matrix $B \cap C^\sigma$, where $\sigma := (\sigma_1, \dots, \sigma_k)$ is an i.i.d. collection of uniform random permutations of $[k]$. Hence, we can couple the two processes X and B together using the Poisson point process P' described above.

Let X evolve according to the Poisson point process P' on $\mathbb{R}^+ \times \prod_{i=1}^k \mathcal{P}_m^{(k)}$ as described above. Let B evolve by the modification that if t is an atom time of P' then we obtain partitions (C_1, \dots, C_k) by sampling $X^i := (X_1^i, X_2^i, \dots)$ i.i.d. from P_i' for each $i = 1, \dots, k$, i.e.

$$\mathbb{P}(X_1^i = j | P_i') = P_j^i,$$

and defining the blocks of C_i as the equivalence classes of X^i . Constructed in this way, $\|C_{ij}\| = P_j^i$ almost surely for each $i, j = 1, \dots, k$ and $(C_1, \dots, C_k) \sim \varrho_\nu^{(k)}$.

After obtaining the C_i , generate, independently of B, C_1, \dots, C_k, P' , i.i.d. uniform permutations $\sigma_1, \dots, \sigma_k$ of $[k]$ and proceed as in the construction of section 4.1 where B, C_1, \dots, C_k are arranged in the matrix $B \cap C^\sigma$ and the blocks of B' are obtained as the marginal column totals of $B \cap C^\sigma$. The (i, j) th entry of $B \cap C^\sigma$ is $C_{i, \sigma_i(j)} \cap B_i$ for which we have $\|C_{i, \sigma_i(j)} \cap B_i\| = \|C_{i, \sigma_i(j)}\| \|B_i\| = x_i P_{\sigma_i(j)}^i$ a.s.

By this construction, $B(t)$ is constructed according to a Poisson point process with the same law as that described in section 4.1, and $B(t)$ possesses ranked asymptotic frequencies which correspond to $X(t)$ almost surely for all $t \geq 0$.

Corollary 5.1. $X(t) := (\Lambda(B(t)), t \geq 0)$ exists almost surely.

5.2. Equilibrium measure

Just as the process $(B(t), t \geq 0)$ on $\mathcal{P}^{(k)}$ converges to a stationary distribution, so does its associated measure-valued process $(X(t), t \geq 0)$ from section 5.1.

Theorem 5.2. *The associated measure-valued process X for a ϱ_ν -Markov process with unique stationary measure $\theta(\cdot; \nu)$ has equilibrium measure $\tilde{\theta}(\cdot; \nu)$, the distribution of the ranked frequencies of a $\theta(\cdot; \nu)$ -partition.*

Proof. Proposition 1.4 in [5] states that if a sequence of exchangeable random partitions converges in law on \mathcal{P} to π_∞ then its sequence of ranked asymptotic frequencies converges in law to $|\pi_\infty|^\downarrow$. Hence, from corollary 4.1 we have that X has equilibrium distribution given by the ranked asymptotic frequencies of a $\theta(\cdot; \nu)$ -partition.

6. The (α, k) -Markov process

Pitman [17] discusses a two-parameter family of infinitely exchangeable random partitions called the (α, θ) process which has finite-dimensional distributions

$$p_n(B; \alpha, \theta) := \frac{(\theta/\alpha)^{\uparrow \#B}}{\theta^{\uparrow n}} \prod_{b \in B} -(-\alpha)^{\uparrow \#b}, \quad (6.1)$$

for (α, θ) satisfying either

- $\alpha = -\kappa < 0$ and $\theta = m\kappa$ for some $m = 1, 2, \dots$, or

- $0 \leq \alpha \leq 1$ and $\theta > -\alpha$.

For $k \in \mathbb{N}$ and $\alpha > 0$, a $(-\alpha, k\alpha)$ partition has finite-dimensional distributions

$$\rho_n(B; \alpha, k) = \frac{k!}{(k - \#B)!} \frac{\prod_{b \in B} \Gamma(\alpha + \#b)/\Gamma(\alpha)}{\Gamma(k\alpha + n)/\Gamma(k\alpha)} \quad (6.2)$$

whose support is $\mathcal{P}_{[n]}^{(k)}$.

The distribution of the ranked asymptotic frequencies of an (α, θ) partition is called the Poisson-Dirichlet distribution with parameter (α, θ) , written $\text{PD}(\alpha, \theta)$.

For notational convenience, introduce the α -permanent [14] of an $n \times n$ matrix K ,

$$\text{per}_\alpha K = \sum_{\sigma \in \mathcal{S}_n} \alpha^{\#\sigma} \prod_{i=1}^n K_{i, \sigma(i)},$$

where $\#\sigma$ is the number of cycles of the permutation σ , and note that when $B \in \mathcal{P}_{[n]}$ is regarded as a matrix,

$$\text{per}_\alpha B = \prod_{b \in B} \text{per}_\alpha B|_b = \prod_{b \in B} \Gamma(\alpha + \#b)/\Gamma(\alpha), \quad (6.3)$$

which allows us to write (6.2) as

$$\rho_n(B; \alpha, k) = \frac{k!}{(k - \#B)!} \frac{\text{per}_\alpha B}{(k\alpha)^{\uparrow n}}, \quad (6.4)$$

where $(\beta)^{\uparrow n} = \beta(\beta + 1) \cdots (\beta + n - 1)$.

We now consider a specific sub-family of reversible ϱ_ν -Markov processes for which the transition probabilities can be written down explicitly. For $k \in \mathbb{N}$ and $\alpha > 0$, let ν be the $\text{PD}(-\alpha/k, \alpha)$ distribution on $\mathcal{P}_m^{(k)}$ and define transition probabilities according to the matrix construction based on ν as in section 3. We call this process the (α, k) -Markov process.

Proposition 6.1. *The (α, k) -Markov process has finite-dimensional transition probabilities*

$$p_n(B, B'; \alpha, k) = \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{\prod_{b' \in B'} \Gamma(\alpha/k + \#(b \cap b'))/\Gamma(\alpha/k)}{\Gamma(\alpha + \#b)/\Gamma(\alpha)} \quad (6.5)$$

$$= \frac{k!}{(k - \#B')!} \frac{\text{per}_{\alpha/k}(B \wedge B')}{\text{per}_\alpha B}. \quad (6.6)$$

Proof. Theorem 3.2 and definition 3.3 from [17] shows that the distribution of $B \sim \varrho_\nu$ where $\nu = \text{PD}(-\alpha/k, \alpha)$ is

$$\rho_n(B; \alpha/k, k) = \frac{k!}{(k - \#B)!} \frac{\text{per}_{\alpha/k} B}{(\alpha)^{\uparrow n}}.$$

Combining this and (3.1) yields (6.5); (6.6) follows from (6.3).

Proposition 6.2. *For each $(\alpha, k) \in \mathbb{R}^+ \times \mathbb{N}$ and $n \in \mathbb{N}$, $p_n(\cdot, \cdot; \alpha, k)$ defined in proposition 6.1 is reversible with respect to (6.2) with parameter (α, k) .*

Proof. Let $\rho_n(\cdot; \alpha, k)$ be the distribution with parameter (α, k) defined in (6.2), and $p_n(\cdot, \cdot; \alpha, k)$ be as defined in (6.5). For any $B, B' \in \mathcal{P}_{[n]}^{(k)}$, it is immediate that

$$\rho_n(B; \alpha, k) p_n(B, B'; \alpha, k) = \rho_n(B'; \alpha, k) p_n(B', B; \alpha, k), \quad (6.7)$$

which establishes reversibility.

Bertoin [4] discusses some reversible EFC processes which have $\text{PD}(\alpha, \theta)$ distribution as their equilibrium measure, for $0 < \alpha < 1$ and $\theta > -\alpha$. Here we have shown reversibility with respect to $\text{PD}(\alpha, \theta)$ for $\alpha < 0$ and $\theta = -m\alpha$ for $m \in \mathbb{N}$.

The construction of the continuous-time process is a special case of the procedure in section 4. The measure-valued process $(X(t), t \geq 0)$ based on the (α, k) -Markov process has unique stationary measure $\text{PD}(-\alpha, k\alpha)$, the distribution of the ranked frequencies of a partition with finite-dimensional distributions as in (6.2) with parameter (α, k) .

7. Discussion

The paths of the ϱ_ν -Markov process discussed above are confined to $\mathcal{P}^{(k)}$. Unlike the EFC-process [2], which has a natural interpretation as a model in certain physical sciences, the ϱ_ν -Markov process has no clear interpretation as a physical model. However, the matrix construction introduced in section 3 leads to transition rates which admit a closed form expression in the case of the (α, k) -Markov process.

The (α, k) class of models could be useful as a statistical model for relationships among statistical units which are known to fall into one of k classes. In statistical work, it is important that any observation has positive probability under the specified model. The (α, k) -process assigns positive probability to all possible transitions and so

any observed sequence of partitions in $\mathcal{P}_{[n]}^{(k)}$ will have positive probability for any choice of $\alpha > 0$. In addition, the model is exchangeable, consistent and reversible, particularly attractive mathematical properties which could have a natural interpretation in certain applications. Future work is intended to explore applications for this model, as well as develop some of the tools necessary for its use in statistical inference.

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